One component of the curvature tensor of a Lorentzian manifold

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Abstract

The holonomy algebra $\mathfrak{g}$ of an $n+2$-dimensional Lorentzian manifold (M,g) admitting a parallel distribution of isotropic lines is contained in the subalgebra $\mathfrak{sim}(n)=(\mathbb{R}\oplus\mathfrak{so}(n))\ltimes\mathbb{R}^n\subset\mathfrak{so}(1,n+1)$. An important invariant of $\mathfrak{g}$ is its $\mathfrak{so}(n)$-projection $\mathfrak{h}\subset\mathfrak{so}(n)$, which is a Riemannian holonomy algebra. One component of the curvature tensor of the manifold belongs to the space $\mathcal{P}(\mathfrak{h})$ consisting of linear maps from $\mathbb{R}^n$ to $\mathfrak{h}$ satisfying an identity similar to the Bianchi one. In the present paper the spaces $\mathcal{P}(\mathfrak{h})$ are computed for each possible $\mathfrak{h}$. This gives the complete description of the values of the curvature tensor of the manifold (M,g). These results can be applied e.g. to the holonomy classification of the Einstein Lorentzian manifolds.

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1. Introduction

The classification of the holonomy algebras of Lorentzian manifolds is achieved only recently [4, 24, 14, 15]. The most interesting case is when a Lorentzian manifold (M,g) admits a parallel distribution of isotropic lines and the manifold is locally indecomposable, i.e. locally it is not a product of a Lorentzian and a Riemannian manifold. In this case the holonomy algebra $\mathfrak{g}$ of (M,g) is contained in the maximal subalgebra $\mathfrak{sim}(n)=(\mathbb{R}\oplus\mathfrak{so}(n))\ltimes\mathbb{R}^n$ of the Lorentzian algebra $\mathfrak{so}(1,n+1)$ preserving an isotropic line (the dimension of M is $n+2$). There is a number of recent physics literature dealing with these manifolds, see e.g. [7, 8, 9, 10, 11, 12, 18, 19, 20, 21]. In particular, in [7, 8, 19] expressed the hope that Lorentzian manifolds with the holonomy algebras contained in $\mathfrak{sim}(n)$ will found many applications in physics, e.g. they are of interest in M-theory and string theory.

In [13, 15] the space $\mathcal{R}(\mathfrak{g})$ of the curvature tensors for each Lorentzian holonomy algebra $\mathfrak{g}$, i.e. the space of values of the curvature tensor of a Lorentzian manifold with the holonomy algebra $\mathfrak{g}$, are described. Similar results in the Riemannian case [2] gives a lot of consequences e.g. for Einstein and Ricci-flat manifolds (we explain them below). One component of the space $\mathcal{R}(\mathfrak{g})$ is

$$\mathcal{P}(\mathfrak{h})=\{P\in(\mathbb{R}^n)^*\otimes\mathfrak{h}|(P(u)v,w)+(P(v)w,u)+(P(w)u,v)=0\text{ for all }u,v,w\in\mathbb{R}^n\},$$

where $\mathfrak{h}\subset\mathfrak{so}(n)$ is the $\mathfrak{so}(n)$-projection of $\mathfrak{g}$, which is the holonomy algebra of a Riemannian manifold, and $(\cdot,\cdot)$ is the inner product on $\mathbb{R}^n$. In the present paper we compute the spaces $\mathcal{P}(\mathfrak{h})$ for each Riemannian holonomy algebra $\mathfrak{h}\subset\mathfrak{so}(n)$ (it is enough to assume that $\mathfrak{h}\subset\mathfrak{so}(n)$ is irreducible).

We introduce the $\mathfrak{h}$-equivariant map

$$\widetilde{\mathrm{Ric}}:\mathcal{P}(\mathfrak{h})\to\mathbb{R}^n,\qquad \widetilde{\mathrm{Ric}}(P)=\sum_{i=1}^n P(e_i)e_i,$$

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where $e_1, ..., e_n$ is an orthogonal basis of $\mathbb{R}^n$. If P is a component of the value of the curvature tensor at a point of a manifold (M, g), then $\widetilde{\mathrm{Ric}}(P)$ is a component of the Ricci tensor at this point. We get the decomposition

$$\mathcal{P}(\mathfrak{h}) = \mathcal{P}_0(\mathfrak{h}) \oplus \mathcal{P}_1(\mathfrak{h}),$$

where $\mathcal{P}_0(\mathfrak{h})$ is the kernel of $\widetilde{\mathrm{Ric}}$ and $\mathcal{P}_1(\mathfrak{h})$ is its orthogonal complement in $\mathcal{P}(\mathfrak{h})$. If $\mathfrak{h} \subset \mathfrak{so}(n)$ is irreducible then $\mathcal{P}_1(\mathfrak{h})$ is either trivial or it is isomorphic to $\mathbb{R}^n$. The spaces $\mathcal{P}(\mathfrak{h})$ for $\mathfrak{h} \subset \mathfrak{u}(\frac{n}{2})$ are found in [24]. In Section 5 we compute the spaces $\mathcal{P}(\mathfrak{h})$ for the rest of the algebras. For these computations we turn to the complexification. We consider the representations of semisimple non-simple Lie algebras and the adjoint representations in a unified way. Then we consider case by case the rest of the representations. In particular we show that $\mathcal{P}_0(\mathfrak{h}) \neq 0$ and $\mathcal{P}_1(\mathfrak{h}) = 0$ exactly for the holonomy algebras $\mathfrak{su}(\frac{n}{2})$, $\mathfrak{sp}(\frac{n}{4})$, $\mathfrak{spin}(7)$ and G_2. Next, $\mathcal{P}_1(\mathfrak{h}) \simeq \mathbb{R}^n$ and $\mathcal{P}_0(\mathfrak{h}) \neq 0$ exactly for the holonomy algebras $\mathfrak{so}(n)$, $\mathfrak{u}(\frac{n}{2})$ and $\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)$. For the rest of the Riemannian holonomy algebras (i.e. for the holonomy algebras of symmetric Riemannian spaces different from $\mathfrak{so}(n)$, $\mathfrak{u}(\frac{n}{2})$ and $\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)$) it holds $\mathcal{P}_1(\mathfrak{h}) \simeq \mathbb{R}^n$ and $\mathcal{P}_0(\mathfrak{h}) = 0$. The result is stated in Table 1 of Section 3. We give the explicit forms of some elements $P \in \mathcal{P}(\mathfrak{h})$ in Section 4.

In [15] the study of the holonomy algebras of Einstein Lorentzian manifolds has been begun. It was not possible to complete this study there, as the spaces $\mathcal{P}(\mathfrak{h})$ were not known. In another paper we will use the results obtained here to complete this classification. Here we consider an example dealing with Einstein manifolds admitting a parallel light-like vector field. Any such manifold is Ricci-flat and its holonomy algebra coincides with $\mathfrak{h} \ltimes \mathbb{R}^n$, where $\mathfrak{h} \subset \mathfrak{so}(n)$ is the (not necessary irreducible) holonomy algebra of a Ricci-flat Riemannian manifold.

Necessary facts from the holonomy theory can be found e.g. in [6, 8, 15, 22].

Finally remark that the elements of $\mathcal{P}(\mathfrak{h})$ also appear as a component of the curvature tensor of a Riemannian supermanifold $(\mathcal{M}, g)$: it can be checked that $\mathrm{pr}_{\mathfrak{so}(T_x\mathcal{M}_{\bar{0}})} \circ R_x(\cdot|_{T_x\mathcal{M}_{\bar{0}}}, \xi) \in \mathcal{P}(\mathrm{pr}_{\mathfrak{so}(T_x\mathcal{M}_{\bar{0}})}\, \mathfrak{g})$ for any fixed $\xi \in T_x\mathcal{M}_{\bar{1}}$. Here $\mathfrak{g}$ is the holonomy algebra of $(\mathcal{M}, g)$ at some point x, $T_x\mathcal{M} = T_x\mathcal{M}_{\bar{0}} \oplus T_x\mathcal{M}_{\bar{1}}$ is the tangent superspace [16, 17]. This gives another motivation to the study of the spaces $\mathcal{P}(\mathfrak{h})$.

2. Preliminaries: the spaces of curvature tensors

Let V be a vector space and $\mathfrak{g} \subset \mathfrak{gl}(V)$ a subalgebra. The vector space

$$\mathcal{R}(\mathfrak{g}) = \{R \in \Lambda^2 V^* \otimes \mathfrak{g} | R(u, v)w + R(v, w)u + R(w, u)v = 0 \text{ for all } u, v, w \in V\}$$

is called *the space of curvature tensors of type* $\mathfrak{g}$. If there is a pseudo-Euclidean metric $(\cdot, \cdot)$ on V such that $\mathfrak{g} \subset \mathfrak{so}(V)$, then any $R \in \mathcal{R}(\mathfrak{g})$ satisfies

$$(R(u, v)z, w) = (R(z, w)u, v) \tag{1}$$

for all $u, v, z, w \in V$. For $R \in \mathcal{R}(\mathfrak{g})$ define its Ricci tensor by

$$\mathrm{Ric}(R)(u, v) = \mathrm{tr}(z \mapsto R(u, z)v),$$

$u, v \in V$. A subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ is called *a Berger algebra* if

$$\mathfrak{g} = \mathrm{span}\{R(u, v) | R \in \mathcal{R}(\mathfrak{g}),\ u, v \in V\},$$

i.e. $\mathfrak{g}$ is spanned by the images of the elements $R \in \mathcal{R}(\mathfrak{g})$. If (M, g) is a pseudo-Riemannian manifold (or, more generally, a manifold M with a torsion-free affine connection ∇) and $\mathfrak{g}$ is its holonomy algebra at a point $x \in M$, then $\mathfrak{g}$ is a Berger algebra, the value R_x of the curvature tensor R of (M, g) at the point x belongs to $\mathcal{R}(\mathfrak{g})$, and the value of the Ricci tensor Ric of (M, g) at the point x coincides with $\mathrm{Ric}(R_x)$. This means that the knowledge of the space $\mathcal{R}(\mathfrak{g})$ impose restrictions on the values of R and Ric and it gives consequences for the geometry of (M, g). Let us look how does it work with the Einstein condition in the Riemannian case. The spaces $\mathcal{R}(\mathfrak{h})$ for the holonomy algebras of Riemannian manifolds $\mathfrak{h} \subset \mathfrak{so}(n)$ are

computed by D. V. Alekseevsky in [2]. Let $\mathfrak{h}\subset\mathfrak{so}(n)$ be an irreducible Riemannian holonomy algebra. The space $\mathcal{R}(\mathfrak{h})$ admits the following decomposition into $\mathfrak{h}$-modules

$$\mathcal{R}(\mathfrak{h})=\mathcal{R}_0(\mathfrak{h})\oplus\mathcal{R}_1(\mathfrak{h})\oplus\mathcal{R}'(\mathfrak{h}),\tag{2}$$

where $\mathcal{R}_0(\mathfrak{h})$ consists of the curvature tensors with zero Ricci tensors, $\mathcal{R}_1(\mathfrak{h})$ consists of tensors annihilated by $\mathfrak{h}$ (this space is zero or one-dimensional), $\mathcal{R}'(\mathfrak{h})$ is the complement to these two spaces. Each element of $\mathcal{R}'(\mathfrak{h})$ has zero scalar curvature and non-zero Ricci tensor. If $\mathcal{R}(\mathfrak{h})=\mathcal{R}_1(\mathfrak{h})$, then any Riemannian manifold with the holonomy algebra $\mathfrak{h}$ is locally symmetric. Such subalgebra $\mathfrak{h}\subset\mathfrak{so}(n)$ is called *a symmetric Berger algebra*. The holonomy algebras of irreducible Riemannian symmetric spaces are exhausted by $\mathfrak{so}(n)$, $u(\frac{n}{2})$, $\mathfrak{sp}(\frac{n}{4})\oplus\mathfrak{sp}(1)$ and by symmetric Berger algebras $\mathfrak{h}\subset\mathfrak{so}(n)$. Note that $\mathcal{R}(\mathfrak{h})=\mathcal{R}_0(\mathfrak{h})$ if $\mathfrak{h}$ is any of the algebras: $\mathfrak{su}(\frac{n}{2})$, $\mathfrak{sp}(\frac{n}{4})$, $G_2\subset\mathfrak{so}(7)$, $\mathfrak{spin}(7)\subset\mathfrak{so}(8)$. This implies that each Riemannian manifold with any of these holonomy algebras is Ricci-flat ($\mathrm{Ric}=0$). Remark that any locally symmetric Riemannian manifold is Einstein ($\mathrm{Ric}=\Lambda g$, $\Lambda\in\mathbb{R}$) and not Ricci-flat. Next, $\mathcal{R}(\mathfrak{u}(\frac{n}{2}))=\mathbb{R}\oplus\mathcal{R}'(\mathfrak{u}(\frac{n}{2}))\oplus\mathcal{R}(\mathfrak{su}(\frac{n}{2}))$ and $\mathcal{R}(\mathfrak{sp}(\frac{n}{4})\oplus\mathfrak{sp}(1))=\mathbb{R}\oplus\mathcal{R}(\mathfrak{sp}(\frac{n}{4}))$. Hence any Riemannian manifold with the holonomy algebra $\mathfrak{sp}(\frac{n}{4})\oplus\mathfrak{sp}(1)$ is Einstein and not Ricci-flat, and a Riemannian manifold with the holonomy algebra $\mathfrak{u}(\frac{n}{2})$ can not be Ricci-flat. Finally, if an indecomposable n-dimensional Riemannian manifold is Ricci-flat, then its holonomy algebra is one of $\mathfrak{so}(n)$, $\mathfrak{su}(\frac{n}{2})$, $\mathfrak{sp}(\frac{n}{4})$, $G_2\subset\mathfrak{so}(7)$, $\mathfrak{spin}(7)\subset\mathfrak{so}(8)$.

Irreducible holonomy algebras $\mathfrak{g}\subset\mathfrak{gl}(n,\mathbb{R})$ of torsion-free affine connections are classified by S. Merkulov and L. Schwachhöfer in [25, 26]. In [3] S. Armstrong, analysing the spaces $\mathcal{R}(\mathfrak{g})$, found which of these holonomy algebras correspond to Ricci-flat connections.

Consider now the case of Lorentzian manifolds. From the Wu Theorem [28] it follows that any Lorentzian manifold (M,g) is either locally a product of the manifold $(\mathbb{R},-(dt)^2)$ and of a Riemannian manifold, or of a Lorentzian and a Riemannian manifold, or it is locally indecomposable, i.e. it does not admit such decompositions. If the manifold (M,g) is simply connected and geodesically complete, then these decompositions are global. This allows us to consider locally indecomposable Lorentzian manifolds. The only irreducible Lorentzian holonomy algebra is the whole Lie algebra $\mathfrak{so}(1,n+1)$ [5] (the dimension of M is $n+2$). If (M,g) is locally indecomposable and its holonomy algebra $\mathfrak{g}$ is different from $\mathfrak{so}(1,n+1)$, then $\mathfrak{g}$ preserves an isotropic line of the tangent space and (M,g) locally admits parallel distributions of isotropic lines. If M is simply connected, then there exists a global parallel distribution of isotropic lines.

Example 1. *Let (M,g) be an $n+2$-dimensional locally indecomposable Lorentzian manifold admitting a parallel light-like vector field X. Let $x\in M$. We identify the tangent space T_xM with the Minkowski space $\mathbb{R}^{1,n+1}$. Let $p\in\mathbb{R}^{1,n+1}$ be the value of X at the point x. Choose a basis $p,e_1,...,e_n,q$ of $\mathbb{R}^{1,n+1}$ with the following non-zero values of g_x: $g_x(p,q)=1$, $g_x(e_i,e_i)=1$. The subalgebra of $\mathfrak{so}(1,n+1)$ preserving p has the form $\mathfrak{so}(n)\ltimes(p\wedge\mathbb{R}^n)$ (we identify $\mathbb{R}^n$ with $\mathrm{span}\{e_1,...,e_n\}$, and $\wedge^2\mathbb{R}^{1,n+1}$ with $\mathfrak{so}(1,n+1)$ such that it holds $(u\wedge v)(z)=(u,z)v-(v,z)u$; in particular, $\wedge^2\mathbb{R}^n$ is identified with $\mathfrak{so}(n)$). Then the holonomy algebra of (M,g) at the point x is contained in the algebra $\mathfrak{h}\ltimes(p\wedge\mathbb{R}^n)$, where $\mathfrak{h}\subset\mathfrak{so}(n)$ is the $\mathfrak{so}(n)$-projection of $\mathfrak{g}$, which is a (not necessary irreducible) Riemannian holonomy algebra [24]. The value R_x satisfies*

$$R_x(p,\cdot)=0,\quad R_x(u,v)=R_0(u,v)+p\wedge(P(u)v-P(v)u),\quad R_x(u,q)=P(u)-p\wedge T(u)$$

for all $u,v\in\mathbb{R}^n$. Here $R_0\in\mathcal{R}(\mathfrak{h})$, $P\in\mathcal{P}(\mathfrak{h})$, and $T\in\mathrm{End}(\mathbb{R}^n)$, $T^*=T$ [13, 15]. In particular, if $\mathfrak{g}=\mathfrak{h}\ltimes(p\wedge\mathbb{R}^n)$, then $\mathcal{R}(\mathfrak{g})\simeq\mathcal{R}(\mathfrak{h})\oplus\mathcal{P}(\mathfrak{h})\oplus\odot^2\mathbb{R}^n$. Thus the only unknown space in this decomposition is $\mathcal{P}(\mathfrak{h})$. The spaces $\mathcal{R}(\mathfrak{g})$ for other Lorentzian holonomy algebras have similar description [13, 15].

3. Main result

Now we begin to study the space $\mathcal{P}(\mathfrak{h})$, where $\mathfrak{h}\subset\mathfrak{so}(n)$ is an irreducible subalgebra. Consider the $\mathfrak{h}$-equivariant map

$$\widetilde{\mathrm{Ric}}:\mathcal{P}(\mathfrak{h})\to\mathbb{R}^n,\qquad \widetilde{\mathrm{Ric}}(P)=\sum_{i=1}^n P(e_i)e_i.$$

This definition does not depend on the choice of the orthogonal basis $e_1, ..., e_n$ of $\mathbb{R}^n$. Denote by $\mathcal{P}_0(\mathfrak{h})$ the kernel of $\widetilde{\mathrm{Ric}}$ and let $\mathcal{P}_1(\mathfrak{h})$ be its orthogonal complement in $\mathcal{P}(\mathfrak{h})$. Thus,

$$\mathcal{P}(\mathfrak{h}) = \mathcal{P}_0(\mathfrak{h}) \oplus \mathcal{P}_1(\mathfrak{h}).$$

Since $\mathfrak{h} \subset \mathfrak{so}(n)$ is irreducible and the map $\widetilde{\mathrm{Ric}}$ is $\mathfrak{h}$-equivariant, $\mathcal{P}_1(\mathfrak{h})$ is either trivial or isomorphic to $\mathbb{R}^n$. The spaces $\mathcal{P}(\mathfrak{h})$ for $\mathfrak{h} \subset \mathfrak{u}(\frac{n}{2})$ are found in [24]. In Section 5 we compute the spaces $\mathcal{P}(\mathfrak{h})$ for the remaining Riemannian holonomy algebras. The result is given in Table 1 (for a compact Lie algebra $\mathfrak{h}$, V_Λ denotes the irreducible representation of $\mathfrak{h}$ given by the irreducible representation of $\mathfrak{h} \otimes \mathbb{C}$ with the highest weight Λ; $((\odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m)_0$ denotes the subspace of $\odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m$ consisting of tensors such that the contraction of the upper index with any down index gives zero).

Table 1. *The spaces $\mathcal{P}(\mathfrak{h})$ for irreducible Riemannian holonomy algebras $\mathfrak{h} \subset \mathfrak{so}(n)$.*

$\mathfrak{h} \subset \mathfrak{so}(n)$	$\mathcal{P}_1(\mathfrak{h})$	$\mathcal{P}_0(\mathfrak{h})$	$\dim \mathcal{P}_0(\mathfrak{h})$
$\mathfrak{so}(2)$	$\mathbb{R}^2$	0	0
$\mathfrak{so}(3)$	$\mathbb{R}^3$	$V_{4\pi_1}$	5
$\mathfrak{so}(4)$	$\mathbb{R}^4$	$V_{3\pi_1+\pi_1'} \oplus V_{\pi_1+3\pi_1'}$	16
$\mathfrak{so}(n)$, $n \geq 5$	$\mathbb{R}^n$	$V_{\pi_1+\pi_2}$	$\frac{(n-2)n(n+2)}{3}$
$\mathfrak{u}(m)$, $n = 2m \geq 4$	$\mathbb{R}^n$	$(\odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m)_0$	$m^2(m-1)$
$\mathfrak{su}(m)$, $n = 2m \geq 4$	0	$(\odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m)_0$	$m^2(m-1)$
$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$, $n = 4m \geq 8$	$\mathbb{R}^n$	$\odot^3(\mathbb{C}^{2m})^*$	$\frac{m(m+1)(m+2)}{3}$
$\mathfrak{sp}(m)$, $n = 4m \geq 8$	0	$\odot^3(\mathbb{C}^{2m})^*$	$\frac{m(m+1)(m+2)}{3}$
$G_2 \subset \mathfrak{so}(7)$	0	$V_{\pi_1+\pi_2}$	64
$\mathfrak{spin}(7) \subset \mathfrak{so}(8)$	0	$V_{\pi_2+\pi_3}$	112
$\mathfrak{h} \subset \mathfrak{so}(n)$, $n \geq 4$, *is a symmetric Berger algebra*	$\mathbb{R}^n$	0	0

Example 2. *In the settings of Example 1 for the value* Ric_x *of the Ricci tensor we have*

$$\mathrm{Ric}_x(p, \cdot) = 0, \quad \mathrm{Ric}_x(u, v) = \mathrm{Ric}(R_0)(u, v),$$
$$\mathrm{Ric}_x(u, q) = g_x(u, \widetilde{\mathrm{Ric}}(P)), \quad \mathrm{Ric}_x(q, q) = \mathrm{tr}\, T.$$

Suppose that (M, g) is an Einstein manifold, i.e. g satisfies the equation of General Relativity in the absence of matter

$$\mathrm{Ric} = \Lambda g, \quad \Lambda \in \mathbb{R}.$$

Using the expression for Ric_x *in [15] it is proved that (M, g) is Ricci-flat (i.e. $\Lambda = 0$) and $\mathfrak{g} = \mathfrak{h} \ltimes (p \wedge \mathbb{R}^n)$. Using the above results for the space $\mathcal{P}(\mathfrak{h})$ it can be shown that $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a Ricci-flat Riemannian manifold, in particular there are decompositions*

$$\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \cdots \oplus \mathbb{R}^{n_s} \oplus \mathbb{R}^{n_{s+1}}, \quad \mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_s \oplus \{0\} \tag{3}$$

such that $\mathfrak{h}_i(\mathbb{R}^{n_j}) = 0$ for $i \neq j$, and each $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ coincides with one of the Lie algebras $\mathfrak{so}(n_i)$, $\mathfrak{su}(\frac{n_i}{2})$, $\mathfrak{sp}(\frac{n_i}{4})$, $G_2 \subset \mathfrak{so}(7)$, $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$.

Note that any Riemannian holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$ admits the decomposition (3) and it holds

$$\mathcal{R}(\mathfrak{h}) = \mathcal{R}(\mathfrak{h}_1) \oplus \cdots \oplus \mathcal{R}(\mathfrak{h}_s), \qquad \mathcal{P}(\mathfrak{h}) = \mathcal{P}(\mathfrak{h}_1) \oplus \cdots \oplus \mathcal{P}(\mathfrak{h}_s).$$

Consider the natural $\mathfrak{h}$-equivariant map

$$\tau : \mathbb{R}^n \otimes \mathcal{R}(\mathfrak{h}) \to \mathcal{P}(\mathfrak{h}), \qquad \tau(u \otimes R) = R(\cdot, u).$$

Theorem 1. *For any irreducible subalgebra $\mathfrak{h}\subset\mathfrak{so}(n)$, $n\geq 4$, the $\mathfrak{h}$-equivariant map $\tau:\mathbb{R}^n\otimes\mathcal{R}(\mathfrak{h})\to\mathcal{P}(\mathfrak{h})$ is surjective. Moreover, $\tau(\mathbb{R}^n\otimes\mathcal{R}_0(\mathfrak{h}))=\mathcal{P}_0(\mathfrak{h})$ and $\tau(\mathbb{R}^n\otimes\mathcal{R}_1(\mathfrak{h}))=\mathcal{P}_1(\mathfrak{h})$.*

Proof. If $\mathfrak{h}\subset\mathfrak{so}(n)$ is a Riemannian holonomy algebra, then the theorem follows from Table 1 and the results from [2]. We claim that the only irreducible subalgebra $\mathfrak{h}\subset\mathfrak{so}(n)$ with $\mathcal{R}(\mathfrak{h})\neq 0$ that is not a Berger algebra is $\mathfrak{sp}(\frac{n}{4})\oplus\mathbb{R}J$, where J is a complex structure on $\mathbb{R}^n$. Similarly, the only irreducible subalgebra $\mathfrak{h}\subset\mathfrak{so}(n)$ with $\mathcal{P}(\mathfrak{h})\neq 0$ that is not spanned by images of the elements from $\mathcal{P}(\mathfrak{h})$ is $\mathfrak{sp}(\frac{n}{4})\oplus\mathbb{R}J$. It is enough to prove the second claim. Let $\mathfrak{h}\subset\mathfrak{so}(n)$ be an irreducible subalgebra with $\mathcal{P}(\mathfrak{h})\neq 0$ and such that $\mathfrak{h}\subset\mathfrak{so}(n)$ is not spanned by images of the elements from $\mathcal{P}(\mathfrak{h})$. If $\mathfrak{h}\not\subset\mathfrak{u}(\frac{n}{2})$, then $\mathfrak{h}\otimes\mathbb{C}$ is semisimple non-simple Lie algebra, and $\mathfrak{h}\otimes\mathbb{C}\subset\mathfrak{so}(n,\mathbb{C})$ is an irreducible subalgebra with $\mathcal{P}(\mathfrak{h}\otimes\mathbb{C})\neq 0$. Let $\mathfrak{h}_1\subset\mathfrak{h}\otimes\mathbb{C}$ be the ideal generated by the images of the elements from $\mathcal{P}(\mathfrak{h}\otimes\mathbb{C})$. Proposition 2 below shows that $\mathfrak{h}\otimes\mathbb{C}\subsetneq\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sp}(\frac{n}{2},\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{C})\subset\mathfrak{h}\otimes\mathbb{C}$. The proof of Proposition 2 implies that $\mathfrak{sl}(2,\mathbb{C})\subset\mathfrak{h}_1$. We get that there is a proper ideal $\mathfrak{h}_2\subset\mathfrak{h}\otimes\mathbb{C}$ such that $\mathfrak{h}\otimes\mathbb{C}=\mathfrak{h}_1\oplus\mathfrak{h}_2$ and $\mathcal{P}(\mathfrak{h}\otimes\mathbb{C})=\mathcal{P}(\mathfrak{h}_1)$. The proof of Proposition 2 shows that this is impossible. Hence, $\mathfrak{h}\subset\mathfrak{u}(\frac{n}{2})$. From the results of [24] it follows that $\mathfrak{h}=\mathfrak{sp}(\frac{n}{4})\oplus\mathbb{R}J$. Finally note that $\mathcal{R}(\mathfrak{sp}(\frac{n}{4})\oplus\mathbb{R}J)=\mathcal{R}(\mathfrak{sp}(\frac{n}{4}))$ and $\mathcal{P}(\mathfrak{sp}(\frac{n}{4})\oplus\mathbb{R}J)=\mathcal{P}(\mathfrak{sp}(\frac{n}{4}))$. $\square$

4. The explicit form of some $P\in\mathcal{P}(\mathfrak{h})$

Using the above results and results from [2], we can now explicitly give the spaces $\mathcal{P}(\mathfrak{h})$ in some cases.

From the results of [24] it follows that $\mathcal{P}(\mathfrak{u}(m))\simeq\odot^2(\mathbb{C}^m)^*\otimes\mathbb{C}^m$. Let us describe this isomorphism in the following way. Let $S\in\odot^2(\mathbb{C}^m)^*\otimes\mathbb{C}^m\subset(\mathbb{C}^m)^*\otimes\mathfrak{gl}(m,\mathbb{C})$. We fix an identification $\mathbb{C}^m=\mathbb{R}^{2m}=\mathbb{R}^m\oplus i\mathbb{R}^m$ and choose a basis $e_1,...,e_m$ of $\mathbb{R}^m$. Define the complex numbers S_{abc}, $a,b,c=1,...,m$ such that $S(e_a)e_b=\sum_c S_{acb}e_c$. It holds $S_{abc}=S_{cba}$. Define a map $S_1:\mathbb{R}^{2m}\to\mathfrak{gl}(2m,\mathbb{R})$ by the conditions $S_1(e_a)e_b=\sum_c\overline{S_{abc}}e_c$, $S_1(ie_a)=-iS_1(e_a)$, and $S_1(e_a)ie_b=iS_1(e_a)e_b$. It is easy to check that $P=S-S_1:\mathbb{R}^{2m}\to\mathfrak{gl}(2m,\mathbb{R})$ belongs to $\mathcal{P}(\mathfrak{u}(n))$ and any element of $\mathcal{P}(\mathfrak{u}(n))$ is of this form. Such element belongs to $\mathcal{P}(\mathfrak{su}(n))$ if and only if $\sum_b S_{abb}=0$ for all $a=1,...,m$, i.e. $S\in(\odot^2(\mathbb{C}^m)^*\otimes\mathbb{C}^m)_0$. If $m=2k$, i.e. $n=4k$, then P belongs to $\mathcal{P}(\mathfrak{sp}(k))$ if and only if $S(e_a)\in\mathfrak{sp}(2k,\mathbb{C})$, $a=1,...,m$, i.e. $S\in(\mathfrak{sp}(2k,\mathbb{C}))^{(1)}\simeq\odot^3(\mathbb{C}^{2k})^*$.

In [15] it is shown that any $P\in\mathcal{P}(\mathfrak{u}(m))$ satisfies $(\widetilde{\mathrm{Ric}}(P),x)=-\operatorname{tr}_{\mathbb{C}}P(Jx)$ for all $x\in\mathbb{R}^{2m}$.

Note that $\mathcal{R}_1(\mathfrak{so}(n))\oplus\mathcal{R}'(\mathfrak{so}(n))\simeq\odot^2\mathbb{R}^n$. Any $R\in\mathcal{R}_1(\mathfrak{so}(n))\oplus\mathcal{R}'(\mathfrak{so}(n))$ is of the form $R=R_S$, where $S:\mathbb{R}^n\to\mathbb{R}^n$ is a symmetric linear map and

$$R_S(x,y)=Sx\wedge y+x\wedge Sy.$$

Similarly, $\mathcal{R}_1(\mathfrak{so}(n))$ is spanned by the element $R=R_{\frac{\mathrm{id}}{2}}$, i.e. $R(x,y)=x\wedge y$. Next, $\tau(\mathbb{R}^n,\mathcal{R}_1(\mathfrak{so}(n))\oplus\mathcal{R}'(\mathfrak{so}(n)))=\mathcal{P}(\mathfrak{so}(n))$. Hence $\mathcal{P}(\mathfrak{so}(n))$ is spanned by the elements P of the form

$$P(y)=Sy\wedge x+y\wedge Sx,$$

where $x\in\mathbb{R}^n$ and $S\in\odot^2\mathbb{R}^n$ are fixed, and $y\in\mathbb{R}^n$ is any vector. For such P it holds $\widetilde{\mathrm{Ric}}(P)=(\operatorname{tr}S-S)x$. We conclude that $\mathcal{P}_0(\mathfrak{so}(n))$ is spanned by the elements P of the form

$$P(y)=Sy\wedge x,$$

where $x\in\mathbb{R}^n$ and $S\in\odot^2\mathbb{R}^n$ satisfy $\operatorname{tr}S=0$ and $Sx=0$, and $y\in\mathbb{R}^n$ is any vector.

The isomorphism $\mathcal{P}_1(\mathfrak{so}(n))\simeq\mathbb{R}^n$ is defined in the following way: $x\in\mathbb{R}^n$ corresponds to $P=x\wedge\cdot\in\mathcal{P}_1(\mathfrak{so}(n))$, i.e. $P(y)=x\wedge y$ for all $y\in\mathbb{R}^n$.

Any $P\in\mathcal{P}_1(\mathfrak{u}(m))$ has the form

$$P(y)=-\frac{1}{2}(Jx,y)J+\frac{1}{4}(x\wedge y+Jx\wedge Jy),$$

where J is the complex structure on $\mathbb{R}^{2m}$, $x\in\mathbb{R}^{2m}$ is fixed, and $y\in\mathbb{R}^{2m}$ is any vector.

Any $P\in\mathcal{P}_1(\mathfrak{sp}(m)\oplus\mathfrak{sp}(1))$ has the form

$$P(y)=-\frac{1}{2}\sum_{\alpha=1}^{3}g(J_\alpha x,y)J_\alpha+\frac{1}{4}\big(x\wedge y+\sum_{\alpha=1}^{3}J_\alpha x\wedge J_\alpha y\big),$$

where (J_1,J_2,J_3) is the quaternionic structure on $\mathbb{R}^{4m}$, $x\in\mathbb{R}^{4m}$ is fixed, and $y\in\mathbb{R}^{4m}$ is any vector.

We will see that for $\mathfrak{h}\subset\mathfrak{so}(\mathfrak{h})$, where $\mathfrak{h}$ is a compact simple Lie algebra any $P\in\mathcal{P}(\mathfrak{h})=\mathcal{P}_1(\mathfrak{h})$ has the form

$$P(y)=[x,y],$$

where $x\in\mathfrak{h}$ is fixed, and $y\in\mathfrak{h}$ is any element.

If $\mathfrak{h}\subset\mathfrak{so}(n)$ is a symmetric Berger algebra, then $\mathcal{P}(\mathfrak{h})=\{R(\cdot,x)|\,x\in\mathbb{R}^n\}$, where R is a generator of $\mathcal{R}(\mathfrak{h})\simeq\mathbb{R}$.

In general, let $\mathfrak{h}\subset\mathfrak{so}(n)$ be an irreducible subalgebra and $P\in\mathcal{P}_1(\mathfrak{h})$. Then $\widetilde{\mathrm{Ric}}(P)\wedge\cdot\in\mathcal{P}_1(\mathfrak{so}(n))$. Furthermore, it is easy to check that $\widetilde{\mathrm{Ric}}\left(P+\frac{1}{n-1}\widetilde{\mathrm{Ric}}(P)\wedge\cdot\right)=0$, that is $P+\frac{1}{n-1}\widetilde{\mathrm{Ric}}(P)\wedge\cdot\in\mathcal{P}_0(\mathfrak{so}(n))$. Thus the inclusion $\mathcal{P}_1(\mathfrak{h})\subset\mathcal{P}(\mathfrak{so}(n))=\mathcal{P}_0(\mathfrak{so}(n))\oplus\mathcal{P}_1(\mathfrak{so}(n))$ is given by

$$P\in P_1(\mathfrak{h})\mapsto\big(P+\frac{1}{n-1}\widetilde{\mathrm{Ric}}(P)\wedge\cdot,-\frac{1}{n-1}\widetilde{\mathrm{Ric}}(P)\wedge\cdot\big)\in\mathcal{P}_0(\mathfrak{so}(n))\oplus\mathcal{P}_1(\mathfrak{so}(n)).$$

This construction defines the tensor $W=P+\frac{1}{n-1}\widetilde{\mathrm{Ric}}(P)\wedge\cdot$, which is the analog of the Weyl tensor for $P\in\mathcal{P}(\mathfrak{h})$, and this tensor is a component of the Weyl tensor of a Lorentzian manifold.

5. Computation of the spaces $\mathcal{P}(\mathfrak{h})$

Let $\mathfrak{h}\subset\mathfrak{so}(n)$ be an irreducible Riemannian holonomy algebra. Since for the subalgebras $\mathfrak{h}\subset\mathfrak{u}(\frac{n}{2})$ the spaces $\mathcal{P}(\mathfrak{h})$ are found in [24], we may assume that $\mathfrak{h}\not\subset\mathfrak{u}(\frac{n}{2})$, then the subalgebra $\mathfrak{h}\otimes\mathbb{C}\subset\mathfrak{so}(n,\mathbb{C})$ is irreducible and it is enough to find the space $\mathcal{P}(\mathfrak{h}\otimes\mathbb{C})$, which equals $\mathcal{P}(\mathfrak{h})\otimes\mathbb{C}$.

Here we compute the spaces $\mathcal{P}(\mathfrak{h})$ for all irreducible Berger subalgebras $\mathfrak{h}\subset\mathfrak{so}(n,\mathbb{C})$. The only non-symmetric Berger subalgebras $\mathfrak{h}\subset\mathfrak{so}(n,\mathbb{C})$ (i.e. subalgebras with $\mathcal{R}(\mathfrak{h})\neq\mathcal{R}_1(\mathfrak{h})$) are $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sp}(2m,\mathbb{C})\subset\mathfrak{so}(4m,\mathbb{C})$ ($m\geq 2$), $\mathfrak{spin}(7,\mathbb{C})\subset\mathfrak{so}(8,\mathbb{C})$ and $G_2^{\mathbb{C}}\subset\mathfrak{so}(7,\mathbb{C})$. The symmetric Berger subalgebras $\mathfrak{h}\subset\mathfrak{so}(m,\mathbb{C})$ (i.e. subalgebras with $\mathcal{R}(\mathfrak{h})=\mathcal{R}_1(\mathfrak{h})$) are given in Table 2 taken from [26].

For simple Lie algebras we use the notation from [27]. For some computations we use the package LiE [23]. Remark that the numbering of the vertices on the Dynkin diagrams for some simple Lie algebras in [27] and [23] are different.

Table 2. *Irreducible symmetric Berger subalgebras* $\mathfrak{h}\subset\mathfrak{so}(m,\mathbb{C})=\mathfrak{so}(V)$.

No.	$\mathfrak{h}$	V
1	$\mathfrak{sp}(2n,\mathbb{C})$, $n\geq 3$	$V_{\pi_2}=\Lambda^2\mathbb{C}^{2n}/\mathbb{C}\omega$
2	$\mathfrak{so}(n,\mathbb{C})$, $n\geq 3$, $n\neq 4$	$V_{2\pi_1}=\odot^2\mathbb{C}^n/\mathbb{C}g$
3	$\mathfrak{h}$ *is a simple Lie algebra*	$\mathfrak{h}$
4	$\mathfrak{so}(9,\mathbb{C})$	$(\Delta_9)^{\mathbb{C}}$
5	$\mathfrak{sp}(8,\mathbb{C})$	$V_{\pi_4}=\Lambda^4\mathbb{C}^8/(\omega\wedge\Lambda^2\mathbb{C}^8)$
6	$F_4^{\mathbb{C}}$	$V_{\pi_1}=\mathbb{C}^{26}$
7	$\mathfrak{sl}(8,\mathbb{C})$	$V_{\pi_4}=\Lambda^4\mathbb{C}^8$
8	$\mathfrak{so}(16,\mathbb{C})$	$(\Delta_{16}^+)^{\mathbb{C}}$
9	$\mathfrak{so}(p,\mathbb{C})\oplus\mathfrak{so}(q,\mathbb{C})$, $p,q\geq 3$	$\mathbb{C}^p\otimes\mathbb{C}^q$
10	$\mathfrak{sp}(2p,\mathbb{C})\oplus\mathfrak{sp}(2q,\mathbb{C})$, $p,q\geq 2$	$\mathbb{C}^{2p}\otimes\mathbb{C}^{2q}$
11	$\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$	$\mathbb{C}^2\otimes\odot^3\mathbb{C}^2$
12	$\mathfrak{sp}(6,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$	$V_{\pi_3}\otimes\mathbb{C}^2=(\Lambda^3\mathbb{C}^6/(\omega\wedge\mathbb{C}^6))\otimes\mathbb{C}^2$
13	$\mathfrak{sl}(6,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$	$V_{\pi_3}\otimes\mathbb{C}^2=\Lambda^3\mathbb{C}^6\otimes\mathbb{C}^2$
14	$\mathfrak{so}(12,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$	$(\Delta_{12}^+)^{\mathbb{C}}\otimes\mathbb{C}^2$
15	$E_7^{\mathbb{C}}\oplus\mathfrak{sl}(2,\mathbb{C})$	$V_{\pi_1}\otimes\mathbb{C}^2=\mathbb{C}^{56}\otimes\mathbb{C}^2$

Lemma 1. *Let V be a real or complex vector space and $\mathfrak{h} \subset \mathfrak{so}(V)$. Then a linear map $R : \Lambda^2 V \to \mathfrak{h}$ belongs to $\mathcal{R}(\mathfrak{h})$ if and only if for each $x \in V$ it holds $R(\cdot, x) \in \mathcal{P}(\mathfrak{h})$.*

Proof. If $R \in \mathcal{R}(\mathfrak{h})$, then the inclusion $R(\cdot, x) \in \mathcal{P}(\mathfrak{h})$ follows from (1) and the Bianchi identity. Conversely, if $R(\cdot, x) \in \mathcal{P}(\mathfrak{h})$ for each $x \in V$, then it is not hard to prove that R satisfies (1), and using this it is easy to see that $R \in \mathcal{R}(\mathfrak{h})$. $\square$

Remark that the above lemma can be also applied for irreducible submodules $U \subset \mathcal{R}(\mathfrak{h})$.

Lemma 2. *Let $\mathfrak{h} \subset \mathfrak{so}(n, \mathbb{C}) = \mathfrak{so}(V)$ be an irreducible subalgebra. Then the decomposition of the tensor product $V \otimes \mathfrak{h}$ into irreducible $\mathfrak{h}$-modules is of the form $V \otimes \mathfrak{h} = kV \oplus (\oplus_\lambda V_\lambda)$, where k is the number of non-zero labels on the Dynkin diagram for the representation of $\mathfrak{h}$ on V, and V_λ are pairwise non-isomorphic irreducible $\mathfrak{h}$-modules that are not isomorphic to V.*

Proof. The number of irreducible submodules isomorphic to the highest weight module V_λ in the product $V \otimes \mathfrak{h}$ is equal to

$$\dim\{v \in \mathfrak{h}_{\lambda-\Lambda} | (\mathrm{ad}_{A_i})^{\Lambda_i+1} v = 0,\ i = 1, ..., l\},$$

where Λ is the highest weight of V, l is the rang of $\mathfrak{h}$, A_i are canonical generators of $\mathfrak{h}$ corresponding to the simple positive roots, and Λ_i are the labels on the Dynkin diagram defining Λ, see e.g. [27]. If $\Lambda \neq \lambda$, then $\dim \mathfrak{h}_{\lambda-\Lambda}$ equals either 0 or 1. This shows that all V_λ are pairwise different. We get that

$$k = \dim\{v \in \mathfrak{h}_0 | (\mathrm{ad}_{A_i})^{\Lambda_i+1} v = 0,\ i = 1, ..., l\},$$

where $\mathfrak{h}_0$ is the fixed Cartan subalgebra of $\mathfrak{h}$. If $\Lambda_i > 0$, then obviously $(\mathrm{ad}_{A_i})^{\Lambda_i+1} v = 0$. We get

$$k = \dim\{v \in \mathfrak{h}_0 | [A_i, v] = 0 \text{ whenever } \Lambda_i = 0\}.$$

The matrix of the obtained homogeneous system of linear equations consists of the lines of the Cartan matrix of $\mathfrak{h}$ corresponding to i with $\Lambda_i = 0$. Since the Cartan matrix is non-degenerate, we immediately get the proof of the lemma. $\square$

Proposition 1. *Let V be a real or complex vector space and $\mathfrak{h} \subset \mathfrak{so}(V)$ be an irreducible subalgebra. If $V \otimes \mathfrak{h}$ contains only one irreducible submodule isomorphic to V, then $\mathcal{P}_1(\mathfrak{h}) \simeq V$ if and only if $\mathcal{R}_1(\mathfrak{h}) \simeq \mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, respectively.*

Proof. Let $\mathrm{Hom}_0(V, \mathfrak{h}) \subset \mathrm{Hom}(V, \mathfrak{h})$ be the subset consisting of the maps $\varphi : V \to \mathfrak{h}$ such that $\sum_{i=1}^n \varphi(e_i)e_i = 0$. Denote by $\mathrm{Hom}_1(V, \mathfrak{h})$ its orthogonal complement, then

$$\mathrm{Hom}(V, \mathfrak{h}) = \mathrm{Hom}_0(V, \mathfrak{h}) \oplus \mathrm{Hom}_1(V, \mathfrak{h}).$$

It is easy to see that $\mathrm{Hom}_1(V, \mathfrak{h}) \simeq V$. Note that $\mathrm{Hom}_1(V, \mathfrak{so}(n)) = \mathcal{P}_1(\mathfrak{so}(n))$.

Lemma 3. $\mathrm{Hom}_1(V, \mathfrak{h}) = \{\mathrm{pr}_{\mathfrak{h}} \circ P | P \in \mathcal{P}_1(\mathfrak{so}(n))\}$.

Let $\varphi \in \mathrm{Hom}_0(V, \mathfrak{h})$ and $P \in \mathcal{P}_1(\mathfrak{so}(n))$, then

$$(\varphi, \mathrm{pr}_{\mathfrak{h}} \circ P) = \sum_{i,j=1}^n (e_i \otimes \varphi(e_i), e_j \otimes \mathrm{pr}_{\mathfrak{h}} \circ P(e_j)) = \sum_{i=1}^n (\varphi(e_i), \mathrm{pr}_{\mathfrak{h}} \circ P(e_i)) = \sum_{i=1}^n (\varphi(e_i), P(e_i)) = (\varphi, P) = 0,$$

where we used the scalar products on different tensor spaces and the fact that $\mathrm{Hom}_0(V, \mathfrak{h}) \subset \mathrm{Hom}_0(V, \mathfrak{so}(n))$ is orthogonal to $\mathrm{Hom}_1(V, \mathfrak{so}(n))$. On the other hand, suppose that $\mathrm{pr}_{\mathfrak{h}} \circ P = 0$. Recall that P is of the form $x_0 \wedge \cdot$ for some $x_0 \in V$. Then $\mathfrak{h}$ annihilates x_0 and we get $x_0 = 0$. The lemma is proved. $\square$

Note that $\odot^2 \mathfrak{h}$ contains $\mathrm{id}_{\mathfrak{h}}$. Moreover, either $\mathcal{R}_1(\mathfrak{h}) = 0$, or $\mathcal{R}_1(\mathfrak{h}) = \mathbb{F}\,\mathrm{id}_{\mathfrak{h}}$. Let $R : \Lambda^2 V \to \mathfrak{h}$ be the extension of $\mathrm{id}_{\mathfrak{h}}$ such that $R|_{\mathfrak{h}^\perp} = 0$, then $R(x, y) = \mathrm{pr}_{\mathfrak{h}}(x \wedge y)$ for all $x, y \in V$. It is clear that either $\mathcal{P}_1(\mathfrak{h}) = 0$, or $\mathcal{P}_1(\mathfrak{h}) = \mathrm{Hom}_1(V, \mathfrak{h})$. Lemma 1 implies that $\mathcal{P}_1(\mathfrak{h}) = \mathrm{Hom}_1(V, \mathfrak{h})$ if and only if $\mathcal{R}_1(\mathfrak{h}) = \mathbb{F}\,\mathrm{id}_{\mathfrak{h}}$. $\square$

Proposition 2. *Let $\mathfrak{h}_1 \subset \mathfrak{gl}(V_1)$ and $\mathfrak{h}_2 \subset \mathfrak{gl}(V_2)$ be irreducible complex subalgebras and $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \subset \mathfrak{so}(V_1 \otimes V_2) = \mathfrak{so}(V)$. If $\mathfrak{h}$ is different from $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sp}(2m,\mathbb{C})$, then $\mathcal{P}_0(\mathfrak{h}) = 0$. Consequently, if $\mathfrak{h}$ is a symmetric Berger algebra, then $\mathcal{P}(\mathfrak{h}) = \mathcal{P}_1(\mathfrak{h}) \simeq V$. Moreover, $\mathcal{P}_1(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sp}(2m,\mathbb{C})) \simeq V$ and $\mathcal{P}_0(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sp}(2m,\mathbb{C})) = (\mathfrak{sp}(2m,\mathbb{C}))^{(1)} \oplus (\mathfrak{sp}(2m,\mathbb{C}))^{(1)}$, where $(\mathfrak{sp}(2m,\mathbb{C}))^{(1)} \simeq \odot^3(\mathbb{C}^{2m})^*$ is the first prolongation of the subalgebra $\mathfrak{sp}(2m,\mathbb{C}) \subset \mathfrak{gl}(2m,\mathbb{C})$.*

Proof. First suppose that the dimensions of V_1 and V_2 are greater then 2. Let $\mathfrak{h}$ be one of the following: $\mathfrak{h} = \mathfrak{so}(V_1) \oplus \mathfrak{so}(V_2)$, $\mathfrak{h} = \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)$.

We claim that there is no $P \in \mathcal{P}(\mathfrak{h})$ taking values either in $\mathfrak{h}_1$ or in $\mathfrak{h}_2$, i.e. $\mathcal{P}(\mathfrak{h}_1 \subset \mathfrak{so}(V)) = \mathcal{P}(\mathfrak{h}_2 \subset \mathfrak{so}(V)) = 0$. Indeed, since the dimensions of V_1 and V_2 are greater then 2, both $\mathfrak{h}_1$ and $\mathfrak{h}_2$ preserve more then two vector subspaces of V and act in these subspaces in the same time. From this it is easily follows that $\mathcal{P}(\mathfrak{h}_1 \subset \mathfrak{so}(V)) = \mathcal{P}(\mathfrak{h}_2 \subset \mathfrak{so}(V)) = 0$. The claim is proved.

Note that $\mathfrak{h} \subset \mathfrak{so}(V)$ is a symmetric Berger subalgebra, and there are exactly two non-zero labels on the Dynkin diagram of $\mathfrak{h}$ defining the representation V. One of these labels is on the Dynkin diagram of $\mathfrak{h}_1$ and the other one is on the Dynkin diagram of $\mathfrak{h}_2$. Hence $V \otimes \mathfrak{h}$ contains two irreducible components isomorphic to V. Next, $V \otimes \mathfrak{h} = (\mathfrak{h}_1 \otimes V_1 \otimes V_2) \oplus (\mathfrak{h}_2 \otimes V_1 \otimes V_2)$. This shows that one irreducible component $V \subset V \otimes \mathfrak{h}$ belongs two $\mathfrak{h}_1 \otimes V_1 \otimes V_2$ and another one belongs to $\mathfrak{h}_2 \otimes V_1 \otimes V_2$. Hence none of them belong two $\mathcal{P}(\mathfrak{h})$. On the other side, $\mathcal{P}(\mathfrak{h})$ contains $\mathcal{P}_1(\mathfrak{h}) \simeq V$. We conclude that $(V \oplus V) \cap \mathcal{P}(\mathfrak{h}) = \mathcal{P}_1(\mathfrak{h}) \simeq V$. If $V_\lambda \subset V \otimes \mathfrak{h}$ is an irreducible $\mathfrak{h}$-submodule not isomorphic to V, then V_λ is contained ether in $\mathfrak{h}_1 \otimes V_1 \otimes V_2$, or in $\mathfrak{h}_2 \otimes V_1 \otimes V_2$, i.e. it is not contained in $\mathcal{P}(\mathfrak{h})$.

If $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \subset \mathfrak{so}(V_1 \otimes V_2) = \mathfrak{so}(V)$ is an irreducible subalgebra different from the above two, then it is properly contained either in $\mathfrak{f} = \mathfrak{so}(V_1) \oplus \mathfrak{so}(V_2)$, or in $\mathfrak{f} = \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)$. Next, $\mathcal{P}(\mathfrak{h}) = (V \otimes \mathfrak{h}) \cap \mathcal{P}(\mathfrak{f})$. Since $\mathcal{P}(\mathfrak{f}) \simeq V$ is an irreducible $\mathfrak{h}$-module and the images of the elements of $\mathcal{P}(\mathfrak{f})$ span $\mathfrak{f}$, we get that $\mathcal{P}(\mathfrak{h}) = 0$.

If $\dim V_1 = 2$, then $\mathfrak{h}_1 = \mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{h}_2 \subset \mathfrak{sp}(2m,\mathbb{C})$ is a proper irreducible subalgebra, $m \geq 2$. Consider the Lie algebra $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sp}(2m,\mathbb{C})$. It is easy to see that $\mathcal{P}(\mathfrak{sp}(2m,\mathbb{C}) \subset \mathfrak{so}(V)) = (\mathfrak{sp}(2m,\mathbb{C}))^{(1)} \oplus (\mathfrak{sp}(2m,\mathbb{C}))^{(1)}$. Using this and the above arguments, we get that $\mathcal{P}_1(\mathfrak{h}) \simeq V$ and $\mathcal{P}_0(\mathfrak{h}) = (\mathfrak{sp}(2m,\mathbb{C}))^{(1)} \oplus (\mathfrak{sp}(2m,\mathbb{C}))^{(1)}$. If $\mathfrak{h}_2 \subset \mathfrak{sp}(2m,\mathbb{C})$ is a proper subalgebra, then $\mathcal{P}_0(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{h}_2) = (V \otimes (\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{h}_2)) \cap \mathcal{P}_0(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sp}(2m,\mathbb{C}))$ and this intersection is zero, since it holds $(\mathfrak{h}_2)^{(1)} = 0$ [26]. The proposition is proved. $\Box$

Let $\mathfrak{h}$ be simple and δ be its highest root. Let $V = V_\Lambda$. The tensor product $V \otimes \mathfrak{h}$ contains the $\mathfrak{h}$-submodule $V_{\Lambda+\delta}$. Let $A_\delta \in \mathfrak{h}$ and $v_\Lambda \in V$ be the highest root and highest weight vectors, respectively. Then $v_\Lambda \otimes A_\delta \in V_{\Lambda+\delta} \subset V \otimes \mathfrak{h}$, and $V_{\Lambda+\delta} \subset \mathcal{P}(\mathfrak{h})$ if and only if $v_\Lambda \otimes A_\delta \in \mathcal{P}(\mathfrak{h})$. It is easy to see that the condition $v_\Lambda \otimes A_\delta \in \mathcal{P}(\mathfrak{h})$ holds if and only if A_δ has rank two and $A_\delta v_{-\Lambda} \neq 0$, where $v_{-\Lambda}$ is the lowest vector in V (note that v_Λ and $v_{-\Lambda}$ are isotropic, and $(v_\Lambda, v_{-\Lambda}) \neq 0$). If A_δ has rank two, then $A_\delta \odot A_\delta \in \mathcal{R}(\mathfrak{h})$ and $\mathfrak{h} \subset \mathfrak{so}(V)$ is a non-symmetric Berger subalgebra. We have proved that if $\mathfrak{h} \subset \mathfrak{so}(V)$ is a symmetric Berger subalgebra or $\mathcal{R}(\mathfrak{h}) = 0$, then $V_{\Lambda+\delta} \cap \mathcal{P}(\mathfrak{h}) = 0$. Thus we need only to consider irreducible submodules $V_\lambda \subset V \otimes \mathfrak{h}$ not isomorphic to $V_{\Lambda+\delta}$ and V (if the representation is given by the Dynkin diagram with only one non-zero label). We will see that such submodules are never contained in $\mathcal{P}(\mathfrak{h})$.

It is easy to get that $\mathbb{C}^n \otimes \mathfrak{so}(n,\mathbb{C}) = \mathbb{C}^n \oplus V_{\pi_1 \oplus \pi_2} \oplus V_{\pi_3}$ $(n \geq 5)$, $\mathbb{C}^7 \otimes G_2^{\mathbb{C}} = \mathbb{C}^7 \oplus V_{\pi_1+\pi_2} \oplus V_{2\pi_2}$ and $\mathbb{C}^8 \otimes \mathfrak{so}(7) = \mathbb{C}^8 \oplus V_{\pi_2+\pi_3} \oplus V_{\pi_1+\pi_3}$.

Lemma 4. *We have $\mathcal{P}_1(\mathfrak{so}(n,\mathbb{C})) \simeq \mathbb{C}^n$, $\mathcal{P}_0(\mathfrak{so}(n,\mathbb{C})) = V_{\pi_1 \oplus \pi_2}$ $(n \geq 5)$, $\mathcal{P}(G_2^{\mathbb{C}}) = \mathcal{P}_0(G_2^{\mathbb{C}}) = V_{\pi_1+\pi_2}$ and $\mathcal{P}(\mathfrak{spin}(7,\mathbb{C})) = \mathcal{P}_0(\mathfrak{spin}(7,\mathbb{C})) = V_{\pi_2+\pi_3}$.*

Proof. From Proposition 1 it follows that $\mathcal{P}_1(\mathfrak{so}(n,\mathbb{C})) \simeq \mathbb{C}^n$, and for both Lie algebras $G_2^{\mathbb{C}} \subset \mathfrak{so}(7,\mathbb{C})$ and $\mathfrak{spin}(7,\mathbb{C}) \subset \mathfrak{so}(8,\mathbb{C})$ it holds $\mathcal{P}_1(\mathfrak{h}) = 0$, i.e. $\mathcal{P}(\mathfrak{h}) \cap V = 0$.

Let $\mathfrak{h} = G_2^{\mathbb{C}}$. We have $\Lambda = \pi_1 = \epsilon_1$ and $\delta = \pi_2 = \epsilon_1 - \epsilon_3$. It is easy to see that $v_\Lambda \otimes A_\delta \in \mathcal{P}(\mathfrak{h})$, i.e. $V_{\Lambda+\delta} \subset \mathcal{P}(\mathfrak{h})$. Next, $\subset V \otimes \mathfrak{h}$ contains a 3-dimensional vector subspace of weight $2\pi_2$. This subspace is spanned by the vectors $v_{\epsilon_1} \otimes A_{\epsilon_1}$, $v_{-\epsilon_2} \otimes A_{\epsilon_1-\epsilon_3}$ and $v_{-\epsilon_3} \otimes A_{\epsilon_1-\epsilon_2}$, where A_μ denotes a non-zero root element in $\mathfrak{h}$ of weight μ. Moreover this subspace has a 1-dimensional intersection with $V_{2\pi_2}$ and a 2-dimensional

intersection with $V_{\pi_1+\pi_2}$. This shows that $V_{2\pi_2} \subset \mathcal{P}(\mathfrak{h})$ if and only if all these three vectors belong to $\mathcal{P}(\mathfrak{h})$. To see that $v_{\epsilon_1} \otimes A_{\epsilon_1} \notin \mathcal{P}(\mathfrak{h})$ it is enough to write down the definition of $P \in \mathcal{P}(\mathfrak{h})$ for the vectors $v_{-\epsilon_1}$, v_{ϵ_2} and v_{ϵ_3}.

The Lie algebras $\mathfrak{so}(n,\mathbb{C})$ and $\mathfrak{spin}(7,\mathbb{C}) \subset \mathfrak{so}(8,\mathbb{C})$ can be considered in the same way. $\square$

Lemma 5. *Let $\mathfrak{h}$ be a simple complex Lie algebra different from $\mathfrak{sl}(2,\mathbb{C})$. Then for the adjoint representation $\mathfrak{h} \subset \mathfrak{so}(\mathfrak{h})$ we have $\mathcal{P}(\mathfrak{h}) = \mathcal{P}_1(\mathfrak{h}) \simeq \mathfrak{h}$.*

Proof. Since the space $\mathcal{R}(\mathfrak{h})$ is one-dimensional and it is spanned by the Lie brackets of $\mathfrak{h}$ [26], any element $P \in \mathcal{P}_1(\mathfrak{h})$ is of the form $P(\cdot) = [\cdot, x]$ for some $x \in \mathfrak{h}$.

Denote by $(\cdot,\cdot)$ the Killing form on $\mathfrak{h}$. Let $P \in \mathcal{P}(\mathfrak{h})$, then for any $x, y, z \in \mathfrak{h}$ it holds

$$\begin{aligned}([P(x), y], z) + ([x, P(y)], z) &= -([P(y), z], x) - ([P(z), x], y) + ([x, P(y)], z)\\ &= -([P(z), x], y) = -(P(z), [x, y]).\end{aligned}$$

Since $\mathfrak{h} \otimes \mathfrak{h} = \odot^2\mathfrak{h} \oplus \Lambda^2\mathfrak{h}$ and the adjoint representation of any simple $\mathfrak{h}$ different from $\mathfrak{sl}(n,\mathbb{C})$ is given by the Dynkin diagram with only one non-zero label, by Lemma 2, we may assume that P is either in $\odot^2\mathfrak{h}$, or in $\Lambda^2\mathfrak{h}$. The same is true for $\mathfrak{h} = \mathfrak{sl}(n,\mathbb{C})$ since it is given by the Dynkin diagram with exactly two non-zero labels, i.e. $\mathfrak{h} \otimes \mathfrak{h}$ contains two irreducible $\mathfrak{h}$-modules isomorphic to $\mathfrak{h}$, one of them coincides with $\mathcal{P}_1(\mathfrak{h})$ and we need to explore the other one.

If $P \in \Lambda^2\mathfrak{h}$, then $P([x, y]) = [P(x), y] + [x, P(y)]$, i.e. P is a derivative of $\mathfrak{h}$. Since $\mathfrak{h}$ is simple, P is of the form $P(\cdot) = [\cdot, x]$ for some $x \in \mathfrak{h}$, i.e. $P \in \mathcal{P}_1(\mathfrak{h})$.

Suppose that $P \in \odot^2\mathfrak{h}$, then $P([x, y]) = -[P(x), y] - [x, P(y)]$. We have

$$\begin{aligned}0 =& P([[x, y], z] + [[y, z], x] + [[z, x], y])\\ =& -[P([x, y]), z] - [[x, y], P(z)] - [P([y, z]), x] - [[y, z], P(x)] - [P([z, x]), y] - [[z, x], P(y)]\\ =& [[P(x), y], z] + [[x, P(y)], z] - [[x, y], P(z)] + [[P(y), z], x] + [[y, P(z)], x] - [[y, z], P(x)]\\ &+ [[P(z), x], y] + [[z, P(x)], y] - [[z, x], P(y)]\\ =& -2([[x, y], P(z)] + [[y, z], P(x)] + [[z, x], P(y)]).\end{aligned}$$

The last equality implies

$$[[P(x), y], z] + [[x, P(y)], z] + [[P(y), z], x] + [[y, P(z)], x] + [[P(z), x], y] + [[z, P(x)], y] = 0.$$

Hence, $[P([x, y]), z] + [P([y, z]), x] + [P([z, x]), y] = 0$, i.e. $P([\cdot,\cdot]) \in \mathcal{R}(\mathfrak{h})$. This shows that $P([x, y]) = c[x, y]$ for all $x, y \in \mathfrak{h}$ and some constant $c \in \mathbb{C}$. We conclude that $P = c\,\mathrm{id}_{\mathfrak{h}}$. It is clear that $P \in \mathcal{P}(\mathfrak{h})$ if and only if $c = 0$. The lemma is proved. $\square$

We are left with the representations 1, 2, 4-8 from Table 2.

For the representations 4, 5, and 8 we prove that $\mathcal{P}_0(\mathfrak{h}) = 0$ using Lemma 1. For each irreducible submodule $V_\lambda \subset V \otimes \mathfrak{h}$ different from the highest one and from V we find a submodule $U \subset \Lambda^2V \otimes \mathfrak{h}$ such that the natural map τ from $(\Lambda^2V \otimes \mathfrak{h}) \otimes V$ to $V \otimes \mathfrak{h}$, mapping $R \otimes x$ to $R(\cdot, x)$, maps $U \otimes V$ onto V_λ. Since $U \not\subset \mathcal{R}(\mathfrak{h})$, we get that $V_\lambda \not\subset \mathcal{P}(\mathfrak{h})$.

The subalgebra $\mathfrak{h} = \mathfrak{so}(9,\mathbb{C}) \subset \mathfrak{so}((\Delta_9)^{\mathbb{C}}) = \mathfrak{so}(16,\mathbb{C})$. It holds $V \otimes \mathfrak{h} = V \oplus V_{\pi_2+\pi_4} \oplus V_{\pi_1+\pi_4}$. The submodule $V_{\pi_2+\pi_4} \subset V \otimes \mathfrak{h}$ is the highest one and we are left with the submodule $V_{\pi_1+\pi_4}$. The $\mathfrak{h}$-module $\odot^2\mathfrak{h} \subset \Lambda^2V \otimes \mathfrak{h}$ contains the submodule $V_{2\pi_1}$. We have $V_{2\pi_1} \otimes V = V_{2\pi_1+\pi_4} \oplus V_{\pi_1+\pi_4}$. Since $V_{2\pi_1+\pi_4} \not\subset V \otimes \mathfrak{h}$ and $\tau(V_{2\pi_1} \otimes V) \neq 0$, we get that $\tau(V_{2\pi_1} \otimes V) = V_{\pi_1+\pi_4}$. Since $V_{2\pi_1} \not\subset \mathcal{R}(\mathfrak{h})$, we conclude that $V_{\pi_1+\pi_4} \not\subset \mathcal{P}(\mathfrak{h})$.

The subalgebra $\mathfrak{h} = \mathfrak{so}(16,\mathbb{C}) \subset \mathfrak{so}((\Delta^+_{16})^{\mathbb{C}}) = \mathfrak{so}(128,\mathbb{C})$. It holds $V \otimes \mathfrak{h} = V \oplus V_{\pi_2+\pi_8} \oplus V_{\pi_1+\pi_7}$. The submodule $V_{\pi_2+\pi_8} \subset V \otimes \mathfrak{h}$ is the highest one. The $\mathfrak{h}$-module $\odot^2\mathfrak{h} \subset \Lambda^2V \otimes \mathfrak{h}$ contains the submodule $V_{2\pi_1}$. We have $V_{2\pi_1} \otimes V = V_{\pi_1+\pi_7} \oplus V_{2\pi_1+\pi_8}$. We conclude that $V_{\pi_1+\pi_7} \not\subset \mathcal{P}(\mathfrak{h})$.

The subalgebra $\mathfrak{h} = \mathfrak{sp}(8,\mathbb{C}) \subset \mathfrak{so}(V_{\pi_4}) = \mathfrak{so}(42,\mathbb{C})$. It holds $V \otimes \mathfrak{h} = V \oplus V_{2\pi_1+\pi_4} \oplus V_{\pi_1+\pi_3}$. The submodule $V_{2\pi_1+\pi_4} \subset V \otimes \mathfrak{h}$ is the highest one. The $\mathfrak{h}$-module $\odot^2\mathfrak{h} \subset \Lambda^2V \otimes \mathfrak{h}$ contains the submodule V_{π_2}. We have $V_{\pi_2} \otimes V = V_{\pi_2+\pi_4} \oplus V_{\pi_1+\pi_3} \oplus V_{\pi_2}$. We conclude that $V_{\pi_1+\pi_3} \not\subset \mathcal{P}(\mathfrak{h})$.

The above trick does not work with the representations 1, 2, 6, 7 from Table 2 and we use the direct computations.

The subalgebra $\mathfrak{h} = \mathfrak{sp}(2n,\mathbb{C}) \subset \mathfrak{so}(V_{\pi_2})$. It holds $V\otimes\mathfrak{h} = V\oplus V_{2\pi_1+\pi_2}\oplus V_{\pi_1+\pi_3}\oplus V_{2\pi_1}$. The submodule $V_{2\pi_1+\pi_2}\subset V\otimes\mathfrak{h}$ is the highest one and we are left with the submodules $V_{\pi_1+\pi_3}$ and $V_{2\pi_1}$. The $\mathfrak{h}$-module $\Lambda^2V\otimes\mathfrak{h}$ contains the submodule $V_{\pi_2+\pi_4}$, in the same time $V_{\pi_2+\pi_4}\otimes V$ contains the submodule $V_{\pi_1+\pi_3}$ and it contains none of the submodules $V_{2\pi_1+\pi_2}$ and $V_{2\pi_1}$. We conclude that $V_{\pi_1+\pi_3}\not\subset\mathcal{P}(\mathfrak{h})$.

Consider the submodule $V_{2\pi_1}$. Let $e_1,...,e_n,e_{-1},...,e_{-n}$ be the standard basis of $\mathbb{C}^{2n}$(such that $\omega(e_i,e_{-i})=1$). The highest vector of the module $V_{2\pi_1}$ equals to

$$\varphi=\sum_{i=2}^n e_1\wedge e_i\otimes(E_{1,i}-E_{n+i,n+1}),$$

where $E_{a,b}$ is the matrix with 1 on the position (a,b) and zeros on the other positions. To find φ, we consider the vector subspace of $V\otimes\mathfrak{h}$ of weight $2\pi_1$ and find a vector (defined up to a constant) annihilated by the generators of $\mathfrak{h}$ corresponding to the simple positive roots. Let $x=e_{-1}\wedge e_{-2}$, $y=e_2\wedge e_3$ and $z=e_{-1}\wedge e_{-3}$. Then $(\varphi(x)y,z)+(\varphi(y)z,x)+(\varphi(z)x,y)=2$. Thus, $\varphi\notin\mathcal{P}(\mathfrak{h})$ and $V_{2\pi_1}\not\subset\mathcal{P}(\mathfrak{h})$.

The subalgebra $\mathfrak{h}=\mathfrak{so}(n,\mathbb{C})\subset\mathfrak{so}(V_{2\pi_1})$. It holds $V\otimes\mathfrak{h}=V\oplus V_{2\pi_1+\pi_2}\oplus V_{\pi_1+\pi_3}\oplus V_{\pi_2}$. The submodule $V_{2\pi_1+\pi_2}\subset V\otimes\mathfrak{h}$ is the highest one and we are left with the submodules $V_{\pi_1+\pi_3}$ and V_{π_2}. The $\mathfrak{h}$-module $\odot^2\mathfrak{h}\subset\Lambda^2V\otimes\mathfrak{h}$ contains the submodule V_{π_4}, and $V_{\pi_4}\otimes V$ contains $V_{\pi_1+\pi_3}$ and it contains none of the submodules $V_{2\pi_1+\pi_2}$ and V_{π_2}. We conclude that $V_{\pi_1+\pi_3}\not\subset\mathcal{P}(\mathfrak{h})$.

Consider the submodule V_{π_2}. Let $e_1,...,e_m,e_{-1},...,e_{-m}$ and $e_1,...,e_m,e_{-1},...,e_{-m},e_0$ be the standard bases of $\mathbb{C}^{2m}$ and $\mathbb{C}^{2m+1}$, respectively (such that $g(e_i,e_{-i})=1$ and $g(e_0,e_0)=1$). The highest vector of the module V_{π_2} equals to

$$\begin{aligned}\varphi=&\sum_{i=3}^m e_1\odot e_i\otimes(E_{2,i}-E_{m+i,m+2})-\sum_{i=3}^m e_2\odot e_i\otimes(E_{1,i}-E_{m+i,m+1}),\quad\text{ if } n=2m,\\ \varphi=&e_1\odot e_0\otimes(E_{2,2m+1}-E_{2m+1,m+2})-e_2\odot e_0\otimes(E_{1,2m+1}-E_{2m+1,m+1})\\ &+\sum_{i=3}^m e_1\odot e_i\otimes(E_{2,i}-E_{m+i,m+2})-\sum_{i=3}^m e_2\odot e_i\otimes(E_{1,i}-E_{m+i,m+1}),\quad\text{ if } n=2m+1.\end{aligned}$$

Taking in the both cases $x=e_{-1}\odot e_{-3}$, $y=e_1\odot e_3$ and $z=e_{-1}\odot e_{-2}$, we get $(\varphi(x)y,z)+(\varphi(y)z,x)+(\varphi(z)x,y)=1$. Thus, $\varphi\notin\mathcal{P}(\mathfrak{h})$ and $V_{\pi_2}\not\subset\mathcal{P}(\mathfrak{h})$.

The subalgebra $\mathfrak{sl}(8,\mathbb{C})\subset\mathfrak{so}(\Lambda^4\mathbb{C}^8)$. We have $V\otimes\mathfrak{g}=V\oplus V_{\pi_1+\pi_4+\pi_7}\oplus V_{\pi_1+\pi_3}\oplus V_{\pi_5+\pi_7}$. The submodule $V_{\pi_1+\pi_4+\pi_7}\subset V\otimes\mathfrak{h}$ is the highest one and we are left with the submodules $V_{\pi_1+\pi_3}$ and $V_{\pi_5+\pi_7}$. Let $e_1,...,e_8$ be the standard basis of $\mathbb{C}^8$. The metric on $V=\Lambda^4\mathbb{C}^8$ is given by the exterior multiplication, $(\omega,\theta)=\omega\wedge\theta=\theta\wedge\omega\in\Lambda^8\mathbb{C}^8\simeq\mathbb{C}$, we assume that $(e_1\wedge e_2\wedge e_3\wedge e_4,e_5\wedge e_6\wedge e_7\wedge e_8)=1$. The highest vector of the submodule $V_{\pi_1+\pi_3}\subset V\otimes\mathfrak{h}$ equals to $\varphi=\sum_{i=1}^5 e_1\wedge e_2\wedge e_3\wedge e_i\otimes E_{1,i}$. Taking $x=e_5\wedge e_6\wedge e_7\wedge e_8$, $y=e_2\wedge e_4\wedge e_5\wedge e_6$, and $z=e_3\wedge e_4\wedge e_7\wedge e_8$, we get $(\varphi(x)y,z)+(\varphi(y)z,x)+(\varphi(z)x,y)=-1$. Hence, $V_{\pi_1+\pi_3}\not\subset\mathcal{P}(\mathfrak{h})$. The symmetry of the Dynkin diagram of $\mathfrak{sl}(8,\mathbb{C})$ implies $V_{\pi_5+\pi_7}\not\subset\mathcal{P}(\mathfrak{h})$.

The subalgebra $\mathfrak{h}=F_4^{\mathbb{C}}\subset\mathfrak{so}(26,\mathbb{C})$. To deal with this representation we use the following description of it form [1]. The Lie algebra $F_4^{\mathbb{C}}$ admits the structure of $\mathbb{Z}_2$-graded Lie algebra: $F_4^{\mathbb{C}}=\mathfrak{so}(9,\mathbb{C})\oplus(\Delta_9)^{\mathbb{C}}$. The representation space $\mathbb{C}^{26}$ is decomposed into the direct sum $\mathbb{C}^{26}=\mathbb{C}\oplus\mathbb{C}^9\oplus(\Delta_9)^{\mathbb{C}}$. The elements of the subalgebra $\mathfrak{so}(9,\mathbb{C})\subset F_4^{\mathbb{C}}$ preserve these components, annihilate $\mathbb{C}$ and act naturally on $\mathbb{C}^9$ and $(\Delta_9)^{\mathbb{C}}$. Elements of $(\Delta_9)^{\mathbb{C}}\subset F_4^{\mathbb{C}}$ take $\mathbb{C}$ and $\mathbb{C}^9$ to $(\Delta_9)^{\mathbb{C}}$ (multiplication by constants and the Clifford multiplication, respectively), and take $(\Delta_9)^{\mathbb{C}}$ to $\mathbb{C}\oplus\mathbb{C}^9$ (the charge conjugation plus the natural map assigning a vector to a pair of spinors). Let $P\in\mathcal{P}(F_4^{\mathbb{C}})$. Decompose it as the sum $P=\varphi+\psi$, where φ and ψ take values in $\mathfrak{so}(9,\mathbb{C})$ and $(\Delta_9)^{\mathbb{C}}$, respectively. The condition $P\in\mathcal{P}(F_4^{\mathbb{C}})$ implies

$$\begin{aligned}&\varphi|_{\mathbb{C}^9}\in\mathcal{P}(\mathfrak{so}(9,\mathbb{C})\subset\mathfrak{so}(9,\mathbb{C})),\quad \varphi|_{(\Delta_9)^{\mathbb{C}}}\in\mathcal{P}(\mathfrak{so}(9,\mathbb{C})\subset\mathfrak{so}((\Delta_9)^{\mathbb{C}})),\\ &\varphi(a)=0,\quad(\psi(a)x,s)+(\psi(x)s,a)=0\\ &(\psi(x)y-\psi(y)x,s)+(\varphi(s)x,y)=0,\quad(\psi(r)s-\psi(s)r,x)+(\varphi(x)r,s)=0.\end{aligned}$$

for all $a\in\mathbb{C}$, $x,y\in\mathbb{C}^9$, and $s,r\in(\Delta_9)^{\mathbb{C}}$.

We will denote the $\mathfrak{h}$-modules by $V^{\mathfrak{h}}_\lambda$ and the $\mathfrak{so}(9,\mathbb{C})$-modules by V_λ. We have $V\otimes\mathfrak{h}=V\oplus V^{\mathfrak{h}}_{\pi_1+\pi_4}\oplus V^{\mathfrak{h}}_{\pi_2}$. The submodule $V^{\mathfrak{h}}_{\pi_1+\pi_4}\subset V\otimes\mathfrak{h}$ is the highest one and we need to explore the module $V^{\mathfrak{h}}_{\pi_2}$. Note that $\dim V^{\mathfrak{h}}_{\pi_2}=273$. The above equalities show that P is uniquely defined by $\psi|_{\mathbb{C}^9\oplus(\Delta_9)^{\mathbb{C}}}$. In particular,

$$\psi|_{(\Delta_9)^{\mathbb{C}}}\in(\Delta_9)^{\mathbb{C}}\otimes(\Delta_9)^{\mathbb{C}}=\mathbb{C}\oplus V_{2\pi_4}\oplus V_{\pi_3}\oplus V_{\pi_2}\oplus V_{\pi_1}$$

defines $\varphi|_{\mathbb{C}^9}\in\mathcal{P}(\mathfrak{so}(9,\mathbb{C})\subset\mathfrak{so}(9,\mathbb{C}))=V_{\pi_1}\oplus V_{\pi_1+\pi_2}$. Hence, $\varphi|_{\mathbb{C}^9}\in V_{\pi_1}=\mathcal{P}_1(\mathfrak{so}(9,\mathbb{C})\subset\mathfrak{so}(9,\mathbb{C}))$. Next,

$$\psi|_{\mathbb{C}^9}\in\mathbb{C}^9\otimes(\Delta_9)^{\mathbb{C}}=V_{\pi_4}\oplus V_{\pi_1+\pi_4}$$

defines $\varphi|_{(\Delta_9)^{\mathbb{C}}}\in\mathcal{P}(\mathfrak{so}(9,\mathbb{C})\subset\mathfrak{so}((\Delta_9)^{\mathbb{C}})=V_{\pi_4}$.

It is clear that $\mathcal{P}_1(\mathfrak{h})$ is given by $\mathbb{C}\oplus V_{\pi_1}\subset(\Delta_9)^{\mathbb{C}}\otimes(\Delta_9)^{\mathbb{C}}$ and by $V_{\pi_4}\subset\mathbb{C}^9\otimes(\Delta_9)^{\mathbb{C}}$ (recall that $\mathbb{C}\oplus V_{\pi_1}\oplus V_{\pi_4}=\mathbb{C}\oplus\mathbb{C}^9\oplus(\Delta_9)^{\mathbb{C}}=V$). The dimensions of the $\mathfrak{so}(9,\mathbb{C})$-modules $V_{2\pi_4}$, V_{π_3} V_{π_2} V_{π_1}, and $V_{\pi_1+\pi_4}$ equal, respectively, 126, 84, 36, 9, and 128. We see that the sum of some of these numbers can not equal $\dim V^{\mathfrak{h}}_{\pi_2}=273$. This shows that $V^{\mathfrak{h}}_{\pi_2}\not\subset\mathcal{P}(\mathfrak{h})$.

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